Voronoi and void statistics for superhomogeneous point processes

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(Received 30 April 2004; published 27 October 2004)

We study the Voronoi and void statistics of superhomogeneous (or hyperuniform) point patterns in which the infinite-wavelength density fluctuations vanish. Superhomogeneous or hyperuniform point patterns arise in one-component plasmas, primordial density fluctuations in the Universe, and jammed hard-particle packings. We specifically analyze a certain one-dimensional model by studying size fluctuations and correlations of the associated Voronoi cells. We derive exact results for the complete joint statistics of the size of two Voronoi cells. We also provide a sum rule that the correlation matrix for the Voronoi cells must obey in any space dimension. In contrast to the conventional picture of superhomogeneous systems, we show that infinitely large Voronoi cells or voids can exist in superhomogeneous point processes in any dimension. We also present two heuristic conditions to identify and classify any superhomogeneous point process in terms of the asymptotic behavior of the void size distribution.

DOI: 10.1103/PhysRevE.70.041105 PACS number(s): 05.20.−y, 61.20.Gy, 61.50.Ah

I. INTRODUCTION

Point patterns are ubiquitous in nature. Examples include those defined by the coordinates of the particles in a many-particle system, such as the molecules of a liquid or crystal, stars of a galaxy, or trees in a forest. Understanding how the number of points fluctuates at a given length scale reveals important structural information about the point pattern. Such local density fluctuations have been studied for a variety of physical systems, including one-component plasmas [1], molecular liquids [2], and the large-scale structure of the universe [3].

Point patterns in which the infinite-wavelength density fluctuations vanish, referred to as *superhomogeneous* [3] or *hyperuniform* [4], are of particular interest to us in the present paper. Regular lattices of points in space are the simplest examples of superhomogeneous point patterns, but such point processes are neither statistically spatially stationary (homogeneous) nor isotropic. Stochastic superhomogeneous point processes and fluctuations have been demonstrated to be very important in a variety of physical contexts, including the study of one component plasmas [5], the evolution of primordial matter density fluctuations in cosmology [3], and the structural properties of jammed configurations of hard spheres systems [4]. It is considerably more difficult to construct point patterns that are statistically stationary and isotropic, although some examples have been identified [1,3,4]. In order to shed light on this problem, our general goal is to understand the statistics of the underlying Voronoi cells associated with the points of stationary and isotropic superhomogeneous point processes in arbitrary space dimension d. A Voronoi cell associated with a given point consists of the region of space closer to this point than to any other point [6].

A first step toward the stated goal is to start by examining stationary superhomogeneous point processes in any dimension (where isotropy is not an issue). Specifically, we analyze such a particular one-dimensional model by studying size fluctuations and correlations of the associated Voronoi cells. We derive exact results for the complete joint statistics of the size of two Voronoi cells. It is additionally shown that infinitely large Voronoi cells can exist in superhomogeneous point processes in any dimension. We also provide a sum rule that the correlation matrix for the Voronoi cells must obey in any space dimension.

II. PRELIMINARIES

Before discussing the details of the model, we recall some general definitions of basic quantities that are used to statistically characterize point processes (for rigorous definitions and analysis see Ref. [7]).

A single realization of a point process is completely determined by the stochastic *microscopic density* function $\hat{n}(x)$, which in d dimension, can be expressed as

$$\hat{n}(\mathbf{x}) = \sum_i \delta(\mathbf{x} - \mathbf{x}_i), \tag{1}$$

where $\delta(\mathbf{x})$ is the usual d-dimensional Dirac delta function, $\mathbf{x_i}$ is the position of the ith point in the system and the sum is over all of the points. The microscopic density has the following integral property:

$$\int_{\mathcal{V}} d^d x \hat{n}(\mathbf{x}) = N[\mathcal{V}],$$

where $\mathcal{V}$ is any measurable set of the space (i.e., the one-dimensional line in the one-dimensional case of interest) and $N[\mathcal{V}]$ is the number of points (particles centers) contained in that set.

The statistics of a point process is completely determined by the infinite set of correlation functions:

$$I_m(\mathbf{x}_1, \ldots, \mathbf{x}_m) \equiv \langle \hat{n}(\mathbf{x}_1) \cdots \hat{n}(\mathbf{x}_m) \rangle,$$

for any integer $m \geq 1$, and where $\langle \cdots \rangle$ indicates the ensemble average over all the possible realizations of the point process.

For definitions of more general correlation functions see Ref. [6]. Clearly, all of the functions I_m are invariant under any permutations of the variables $\mathbf{x}_1,\ldots,\mathbf{x}_m$. For $n=1$, we have that

$$I_1(\mathbf{x}) \equiv n(\mathbf{x}) = \langle \hat{n}(\mathbf{x}) \rangle$$

gives the *local average density* of points at the spatial position $\mathbf{x}$ and characterize completely all the one-point statistical properties of the system. However, very often a constant *global* average density is also evaluated through a volume average,

$$n_0 = \lim_{V\to+\infty} \frac{1}{V} \int_V d^d x \hat{n}(\mathbf{x}), \tag{2}$$

which gives the average density of particles in the system as a whole, and where V is, for example, a spherical volume. Systems in which *volume* averages as in Eq. (2) are equal to the relative *ensemble* averages are called *ergodic systems*. The quantity $I_2(\mathbf{x},\mathbf{y})d^d x\, d^d y$ gives the joint *a priori* probability of finding a point in the volume element $d^d x$ around $\mathbf{x}$ and at the same time another in the element $d^d y$ around $\mathbf{y}$. It is the most commonly used function to study the correlation properties of an empirical particle distribution.

If all the $I_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ are invariant under a constant translation of all the points, i.e., if $I_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)=I_m(\mathbf{x}_1+\mathbf{x}_0,\ldots,\mathbf{x}_m+\mathbf{x}_0)$ for any $\mathbf{x}_0$ and m, the stochastic point process is said to be *spatially statistically stationary* (or *statistically homogeneous*). In most of what follows we will limit our considerations to this class of point process. In such instances, $n(\mathbf{x})=n_0>0$ (the condition >0 excludes fractal point distributions) does not depend on $\mathbf{x}$, and $I_2(\mathbf{x},\mathbf{y})=I_2(\mathbf{x}-\mathbf{y})$ depends only on the displacement vector. If moreover the system is statistically isotropic I_2 depends only on the scalar distance $|\mathbf{x}-\mathbf{y}|$.

A d-dimensional point process is said to be *ergodic* if, for any function $F[\hat{n}(\mathbf{x}_1),\hat{n}(\mathbf{x}_2),\ldots,\hat{n}(\mathbf{x}_l)]$ of the microscopic density $\hat{n}(\mathbf{x})$ in the arbitrary points $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_l$ (where l is finite but arbitrary), the following relation holds:

$$\lim_{V\to+\infty} \frac{1}{V} \int_V d^d x_0\, F[\hat{n}(\mathbf{x}_0+\mathbf{x}_1),\hat{n}(\mathbf{x}_0+\mathbf{x}_2),\ldots,\hat{n}(\mathbf{x}_0+\mathbf{x}_l)] = \langle F[\hat{n}(\mathbf{x}_1),\hat{n}(\mathbf{x}_2),\ldots,\hat{n}(\mathbf{x}_l)]\rangle. \tag{3}$$

It is clear from Eq. (3) that spatial stationarity is a necessary condition for ergodicity [6,8,9]. Ergodicity is often supposed *a priori* as a valid working hypothesis in the analysis of (spatially or temporally) stationary stochastic processes.

In order to measure the density-fluctuation correlations $\delta\hat{n}(\mathbf{x})=\hat{n}(\mathbf{x})-n_0$ between two different points in a statistically stationary point process, the *covariance* function (also called *reduced* two-point correlation function) $C(\mathbf{x})$ is introduced via

$$C(\mathbf{x}) = \langle \delta\hat{n}(\mathbf{x}_0)\delta\hat{n}(\mathbf{x}_0+\mathbf{x})\rangle = I_2(\mathbf{x}) - n_0^2. \tag{4}$$

It is simple to show, from (1) that $C(\mathbf{x})$ can be written as

$$C(\mathbf{x}) = n_0\delta(\mathbf{x}) + n_0^2 h(\mathbf{x}),$$

where $n_0\delta(\mathbf{x})$ is the *diagonal* part of $C(\mathbf{x})$ present in any stochastic point process independently of the correlations between different spatial points and due only to the *discrete* nature of the *massive* point-particle distribution, while $n_0^2 h(\mathbf{x})$, meaningful for $x\neq 0$, is the *nondiagonal* part characterizing the real correlation between different points and vanishing for $x=|\mathbf{x}|\to\infty$. The function $h(\mathbf{x})$ is referred to as the *total correlation function* in the theory of liquids [10].

Another important quantity, characterizing the relative weight of each Fourier mode to a realization of the stochastic point process, is the so-called *power spectrum* $s(\mathbf{k})$ (proportional to the so-called structure factor [11] and called also Bartlett spectrum [12]), which is defined by

$$s(\mathbf{k}) = \lim_{L\to+\infty} \langle |\delta_n(\mathbf{k};L)|^2\rangle, \tag{5}$$

where

$$\delta_n(\mathbf{k};L) = \frac{1}{L^{d/2}} \int_{-L/2}^{L/2} \cdots \int_{-L/2}^{L/2} d^d x\ \delta\hat{n}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

is the Fourier element of the density contrast $\delta\hat{n}(\mathbf{x})$ in a cubic volume of size L. It is simple to show that if the point process is spatially stationary then $s(\mathbf{k})$ is simply the Fourier transform of $C(\mathbf{x})$,

$$s(\mathbf{k}) = n_0 + n_0^2 \int d^d x\ h(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} = n_0 + n_0^2 \hat{h}(\mathbf{k}),$$

where $\hat{h}(\mathbf{k})$ is the Fourier transform in the infinite volume of $h(\mathbf{x})$. This result implies the so-called Wiener-Khinchtine theorem [6,13], which states that the covariance function of a stationary point process has a positive Fourier transform converging to n_0 for sufficiently large k and integrable around $k=0$.

Finally if the system is also statistically isotropic also $s(\mathbf{k})$ depends only on $k=|\mathbf{k}|$.

III. SUPERHOMOGENEOUS (HYPERUNIFORM) POINT PROCESSES

Here we briefly review definitions and basic properties of superhomogeneous (or hyperuniform) point processes. Given a spatially stationary point process in d dimensions, we can define the variance in the number of points in a sphere $\Omega(R)$ of radius R (the origin of the sphere is arbitrary because of the spatial stationarity) as

$$\sigma^2(R) = \langle N^2(R)\rangle - \langle N(R)\rangle^2, \tag{6}$$

where

$$N(R) = \int_{\Omega(R)} d^d x \hat{n}(\mathbf{x}),$$

is the number of points in the sphere $\Omega(R)$, which is a stochastic function.

It is simple to show that Eq. (6) can be expressed in terms of the covariance function $C(\mathbf{x})$ as follows:

$$\sigma^2(R) = \int_{\Omega(R)} \int_{\Omega(R)} d^d x \; d^d y C(\mathbf{x} - \mathbf{y}). \tag{7}$$

Equivalently, we can express the same quantity in terms of the power spectrum $s(\mathbf{k})$ [4] in the following way:

$$\sigma^2(R) = \frac{1}{(2\pi)^d} \int d^d k |w(\mathbf{k};R)|^2 s(\mathbf{k}), \tag{8}$$

where the integral is over all the k space, and

$$w(\mathbf{k};R) = \int_{\Omega(R)} d^d x \; e^{-i\mathbf{k}\cdot\mathbf{x}}$$

is the so-called *window function*.

All stationary point processes can be classified in terms of the scaling behavior of $\sigma^2(R)$ for large R as follows [3].

(i) If

$$\int d^d x \; C(\mathbf{x}) = s(0) = A > 0,$$

i.e., the correlations are mainly positive and short ranged, then

$$\sigma^2(R) \sim R^d$$

for sufficiently large R (i.e., for R larger than the range of correlations). The prototypical example of this class of systems is the so-called Poisson point process [4,14], which can be generated by randomly placing points in the space with a given average density $n_0 > 0$ in an uncorrelated manner. In this case, it is simple to show that simply $C(\mathbf{x}) = n_0 \delta(\mathbf{x})$ and $s(\mathbf{k}) = n_0$. For this reason we call this class of point patterns *essentially Poissonian*. This is the most common behavior for the number fluctuations for homogeneous systems in thermal equilibrium (e.g., an ordinary gas in equilibrium at high temperature or a liquid away from critical points).

(ii) If, instead,

$$\int d^d x \; C(\mathbf{x}) = s(0) = +\infty,$$

with $s(\mathbf{k}) \sim k^{-\gamma}$ for sufficiently small k where, for definiteness, $0 < \gamma < d$, then

$$\sigma^2(R) \sim R^{d+\gamma}$$

for sufficiently large R. In this case, two-point correlations are again mainly positive but are long ranged. This situation characterizes order parameters of a thermodynamical system at the critical point of a second order phase transition (e.g., the gas-liquid transition at the critical temperature and pressure). For this reason, we call this class *critical systems*.

(iii) Finally, if

$$\int d^d x \; C(\mathbf{x}) = s(0) = 0, \tag{9}$$

it is possible to show that

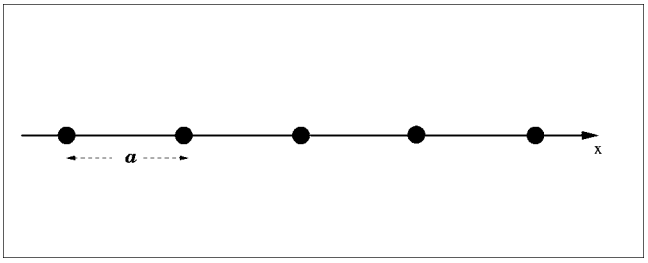

FIG. 1. Schematic representation of the one-dimensional lattice with lattice constant a.

$$\sigma^2(R) \sim R^\alpha, \tag{10}$$

with $\alpha < d$. In particular, it is possible to show that in any case $d-1 \leqslant \alpha < d$, i.e., $\sigma^2(R) \sim R^{d-1}$ is the minimal scaling behavior for the number fluctuations versus R for any point process (all these considerations can be directly extended to include also any "genuine" continuous stochastic mass density field [3]). In this case, there is an exact balance between positive and negative correlations in the density fluctuations in such a way to have Eq. (9). Therefore, infinite wavelength density fluctuations vanish, which imparts a degree of "order" even to stochastic point processes that satisfy (9). At sufficiently small k, we have

$$s(\mathbf{k}) \sim k^\gamma, \tag{11}$$

with $\gamma > 0$. It is possible to show [3] that α and γ are related in the following way: (i) if $0 < \gamma \leqslant 1$, we have $\alpha = d - \gamma$; (ii) if $\gamma \geqslant 1$, then $\alpha = d - 1$ (the "proper" condition for superhomogeneity). For $\gamma = 1$ there will be logarithmic corrections.

Since for the class of systems that satisfy (9) the number fluctuations increase with the spatial scale slower than in a large class of correlated and uncorrelated point processes (e.g., Poisson distribution), we call them superhomogeneous or hyperuniform point processes. Note that superhomogeneous point processes are at a type of "critical" point, but one in which the *direct two-point correlation function* [4] rather than the covariance $C(\mathbf{x})$ is long ranged.

IV. THE ONE-DIMENSIONAL MODEL

In order to construct a superhomogeneous point process suitable for a complete study, we begin with a one-dimensional regular lattice of points, i.e., a chain of point particles with constant spatial separation (lattice constant) a (see Fig. 1). The microscopic density for such a regular point process is given by

$$\hat{n}(x) = \sum_{j=-\infty}^{+\infty} \delta(x - ja), \tag{12}$$

where $a > 0$ is the lattice spacing. Clearly, such a set is not spatially stationary, but only possesses discrete translational invariance. However, it is the one-dimensional superhomogeneous point process with the lowest number variance as a function of R [4]. The global average density of the system is simply $n_0 = 1/a$.

In order to obtain a stochastic superhomogeneous one-dimensional point process suitable for our study, we shuffle

the lattice by introducing a random displacement field. That is, we move each point from its initial lattice position through a random displacement with a given probability distribution, each point being displaced independently of the others. In practice, if the initial position of the mth point is ma, the final one will be $x_m = ma + u_m$, where u_m is a random variable extracted from the probability density function (PDF) $p(u)$. Note that the average density n_0 is not changed by the application of the displacements, since the number of points in the system is conserved.

It is possible to show (see the Appendix) that if each point of a general initial spatial distribution is displaced from its initial position independently of the others with a PDF $p(u)$, then the *new* power spectrum $s(k)$ will be given by

$$s(k) = n_0[1 - |\tilde{p}(k)|^2] + s_I(k)|\tilde{p}(k)|^2, \tag{13}$$

where $s_I(k)$ is the initial power spectrum before the displacements and

$$\tilde{p}(k) = \int_{-\infty}^{+\infty} du\ p(u) e^{-iku} \tag{14}$$

is the Fourier transform of $p(u)$, i.e., the so-called *characteristic function* of the random-displacement PDF (for a more general discussion of the effect of a stochastic displacement field with arbitrary spatial correlation on a given point process see Ref. [15]). In general, we take $p(u)$ to be symmetric, i.e., $p(u) = p(-u)$. Note that for all possible $p(u)$, we have the limit condition $\tilde{p}(0) = 1$ and that for small k in the symmetric case

$$\tilde{p}(k) \simeq 1 - Ak^{\alpha} \tag{15}$$

with $\alpha = 2$ and $A = \overline{u^2}/2$ if $\overline{u^2}$ is finite, and where $\overline{f(u)} = \int_{-\infty}^{+\infty} du\ p(u) f(u)$ means the average over the uncorrelated displacements. For $d = 1$, this is the case if $p(u)$ decreases faster than $|u|^{-3}$ for large $|u|$. Otherwise [15], if $\overline{u^2} = +\infty$, i.e., $p(u) \simeq B|u|^{-\beta}$ for large $|u|$ with $1 < \beta < 3$, then $\alpha = \beta - 1$,

$$A = 2B \int_0^{+\infty} dx\ x^{-\beta}(1 - \cos x), \tag{16}$$

where B is a positive constant.

In the case of a lattice, it is well known [3] and simple to show that

$$s_I(k) = \frac{2\pi}{a^2} \sum_{m \neq 0} \delta\left(k - \frac{2\pi m}{a}\right),$$

where the sum is over all of the integers m, except $m = 0$. Therefore, from Eq. (13), the power spectrum of the "shuffled" lattice is

$$s(k) = \frac{1 - |\tilde{p}(k)|^2}{a} + \frac{2\pi}{a^2} \sum_{m \neq 0} \delta\left(k - \frac{2\pi m}{a}\right) \left| \tilde{p}\left(\frac{2\pi m}{a}\right) \right|^2. \tag{17}$$

Recall that superhomogeneity (or hyperuniformity) of the point process is given by only the behavior of $s(k)$ in the vicinity of $k = 0$. Therefore, since in the *first Brillouin zone* the power spectrum of a lattice is identically zero (i.e., the first Bragg peaks are at $|k| = 2\pi/a$), the small k behavior of $s(k)$ is determined only by that of $\tilde{p}(k)$. In particular, for $|k| < 2\pi/a$ and $|k| \ll (1/A)^{1/\alpha}$ [cf. Eqs. (15) and (16)] we have from the discussion above that

$$s(k) = \frac{2Ak^{\alpha}}{a} \text{ with } \begin{cases} \alpha = 2 \text{ and } A = \dfrac{\overline{u^2}}{2} & \text{if } \overline{u^2} < +\infty \\ \alpha = \beta - 1 \text{ and } A \text{ from Eq.(16)} & \text{if } \overline{u^2} = +\infty \end{cases}, \tag{18}$$

which always satisfies the superhomogeneity condition $\alpha > 0$. In particular, for $\beta > 2$ we have $1 < \alpha \leqslant 2$, and the condition of *minimal* mass fluctuations-length scaling for point process in d dimensions [i.e., $\sigma^2(R) \sim R^{d-1}$] is satisfied (for $\beta = 2$ there are logarithmic corrections in L).

In the case in which each point is completely randomly displaced inside its own unit cell, i.e.,

$$p(u) = \frac{\theta\left(\frac{a}{2} - |u|\right)}{a},$$

where $\theta(x)$ is the usual Heaviside step function, the final point distribution is not only superhomogeneous, but also completely statistically stationary (i.e., with a complete statistical translational invariance), even though the original lattice array was not.

We first analyze the behavior of the fluctuations associated with the volumes of the Voronoi cells in the simple case in which (see Fig. 2)

$$p(u) = \frac{\theta\left(\frac{\Delta}{2} - |u|\right)}{\Delta} \text{ with } \Delta \leqslant a. \tag{19}$$

The statistics of the Voronoi cells are relatively simple because no point is allowed to move into the unit cell centered at the initial position of another point. In what follows, starting from the results for this model, we will extend some of the results to the most general class of superhomogeneous point processes in any dimension.

V. VORONOI-CELL STATISTICS

As stated above, we start from a regular lattice of points with microscopic density given by Eq. (12), and displace each point independently of the others by applying to it a displacement whose PDF $p(u)$ is given by Eq. (19). Taking the Fourier transform of this PDF yields the characteristic function $\tilde{p}(k)$ to be exactly given by

$$\tilde{p}(k) = \frac{\sin\left(\frac{k\Delta}{2}\right)}{\frac{k\Delta}{2}}.$$

Consequently, applying Eq. (17) we obtain

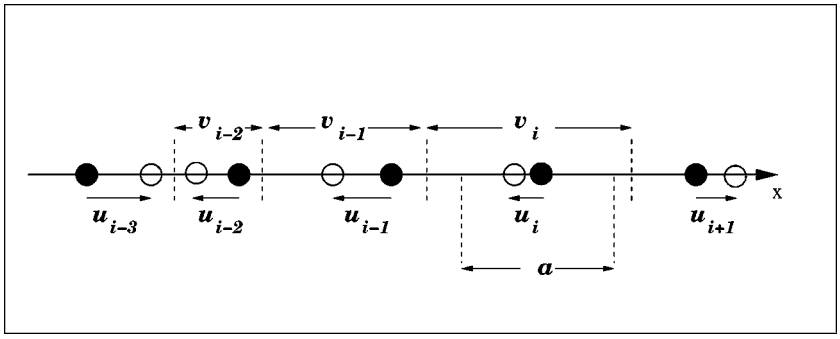

FIG. 2. "Shuffled" lattice with the PDF of the uncorrelated displacements $p(u)=[\theta(\Delta/2)-|u|]/\Delta$with $\Delta \leqslant a$. The filled circles represent the initial lattice configuration (i.e., a lattice with a lattice constant a), while the empty circles are the new positions of the points after the displacements u_i. The quantity v_i is the size of the final Voronoi cell of the point initially at the lattice position $a \cdot i$.

$$s(k) = \frac{1}{a}\left\{1 - \left[\frac{2}{k\Delta}\sin\left(\frac{k\Delta}{2}\right)\right]^2\right\} + \frac{2\pi}{a^2}\sum_{m\neq 0}\delta\left(k - \frac{2\pi m}{a}\right)\left[\frac{a}{m\pi\Delta}\sin\left(\frac{m\pi\Delta}{a}\right)\right]^2. \tag{20}$$

We can verify directly that, since $\sin(m\pi)=0$ for any integer m, only if $\Delta=a$ the contribution to Eq. (20) coming from the Bragg peaks of the underlying lattice structure completely vanishes. In fact, it is the only case in which the point process is fully translationally invariant.

We can now proceed to the evaluation of the statistics of the Voronoi cells. For a point process in any dimension, the Voronoi cell associated with a given point consists of the region of space closer to this point than to any other point. The collection of all of the Voronoi cells that tiles the space is referred to as a Voronoi *tessellation*. Clearly, in the initial lattice configuration, the Voronoi cell associated with each point coincides with the unit cell of size (length) a around each point. According to Eq. (19), a randomly displaced point that was at the original lattice position ja (integer j) remains within its original unit cell. Consequently, we will always refer to this as point j. The size of its new Voronoi cell v_j will be given, by definition, by the size of the line segment that joins the point that lies exactly midway between the points $j+1$ and j and the point lies exactly midway between the points j and $j-1$, i.e.,

$$v_j = a + \frac{u_{j+1} - u_{j-1}}{2}, \tag{21}$$

where u_j is the displacement applied to point j. The PDF $f_1(v)$ characterizing the size of the single Voronoi cell is formally given by

$$f_1(v) = \int\int_{-\infty}^{+\infty} dx\, dy\, p(x)p(y) \times \delta\left(v - a - \frac{x-y}{2}\right). \tag{22}$$

Use of Eq. (19) yields (see Fig. 3)

$$f_1(v) = \frac{2}{\Delta^2} \times \begin{cases} 0 & \text{if } |v-a| \geqslant \frac{\Delta}{2}, \\ 2(v-a)+\Delta & \text{if } -\frac{\Delta}{2} \leqslant v-a \leqslant 0, \\ -2(v-a)+\Delta & \text{if } 0 \leqslant v-a \leqslant \frac{\Delta}{2}. \end{cases} \tag{23}$$

Let $\overline{(\ldots)}$ denote the average over the realizations of the displacement field. Since we start from a deterministic point distribution (i.e., a lattice), this average is equivalent to the ensemble average over the final point process. In general, when also the initial state is a realization of a stochastic point process, the ensemble average over the final configurations by the double average $\langle\overline{(\cdots)}\rangle$ must be taken, where $\langle\cdots\rangle$ is the average over the realizations of the initial point process, and $\overline{(\cdots)}$ is the average over the displacements *conditioned* to the initial configuration. If the realization of the displacement field, seen as a continuous stochastic field with a value $u(\mathbf{x})$ in each spatial point, is independent of the realization of the initial point distribution, the order of the two averages is totally arbitrary. It is only under this hypothesis that Eq. (13) is valid.

Clearly, the average size of a Voronoi cell is given by

$$\overline{v} \equiv \int_0^\infty dv\, v f_1(v) = a.$$

The variance of the size of the Voronoi cell is given by

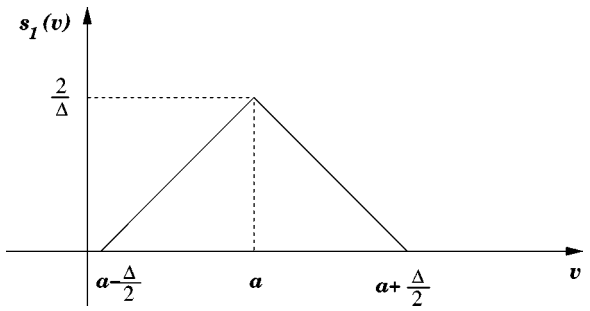

FIG. 3. Representation of the one-cell size PDF $f_1(v)$ for our model.

$$\overline{v^2} - \overline{v}^2 = \frac{\Delta^2}{24}.$$

Note that, as only finite up to $\Delta/2$ jumps are permitted, only finite fluctuations for v are possible. The interesting question of whether infinitely large cell-size fluctuations are permitted in a superhomogeneous point process will be tackled in the next section together with other important aspects of Voronoi cells fluctuations.

In the rest of this section, we analyze the joint probability distribution of two different Voronoi cells. In particular, we find an important "conservation law" for cell-cell correlations.

In order to find the two-cell joint PDF $f_2(v_i, v_j)$, it is important to note that in light of Eq. (21) v_i and v_j are two dependent variables only if $|i-j|=2$. This means that for $|i-j| \neq 2$, we have

$$f_2(v_i, v_j) = f_1(v_i) f_1(v_j).$$

For $j=i+2$, the PDF $f_2(v_i, v_{i+2})$ will be given by the integral

$$f_2(v_i, v_{i+2}) = \int\int\int_{-\infty}^{+\infty} du_{i-1}\, du_{i+1}\, du_{i+3}\, p(u_{i-1}) \times p(u_{i+1}) p(u_{i+3}) \delta\left(v_i - a - \frac{u_{i+1} - u_{i-1}}{2}\right) \times \delta\left(v_{i+2} - a - \frac{u_{i+3} - u_{i+1}}{2}\right), \tag{24}$$

where $p(u)$ is still given by Eq. (19). By performing explicitly the calculations and calling $w_j = v_j - a$ for all j, it is simple to show that

$$f_2(v_i, v_{i+2}) = \frac{4}{\Delta^3} \times \begin{cases} \Delta - 2(w_i + w_{i+2}) & \text{in} \quad A_1, \\ \Delta - 2w_i & \text{in} \quad A_2, \\ \Delta - 2w_{i+2} & \text{in} \quad A_3, \\ \Delta + 2(w_i + w_{i+2}) & \text{in} \quad A_4, \\ \Delta + 2w_i & \text{in} \quad A_5, \\ \Delta - 2w_{i+2} & \text{in} \quad A_6, \\ 0 & \text{elsewhere}, \end{cases} \tag{25}$$

where (see Fig. 4) the A_i are the joint conditions

$$A_1 = \{w_i \geqslant 0 \text{ and } 0 \leqslant w_{i+2} \leqslant -w_i + \Delta/2\},$$

$$A_2 = \{0 \leqslant w_i \leqslant \Delta/2 \text{ and } -w_i \leqslant w_{i+2} \leqslant 0\},$$

$$A_3 = \{w_i \geqslant 0 \text{ and } -\Delta/2 \leqslant w_{i+2} \leqslant -w_i\},$$

$$A_4 = \{w_i \leqslant 0 \text{ and } -w_i - \Delta/2 \leqslant w_{i+2} \leqslant 0\},$$

$$A_5 = \{-\Delta/2 \leqslant w_i \leqslant 0 \text{ and } 0 \leqslant w_{i+2} \leqslant -w_i\},$$

$$A_6 = \{w_i \leqslant 0 \text{ and } -w_i \leqslant w_{i+2} \leqslant \Delta/2\}.$$

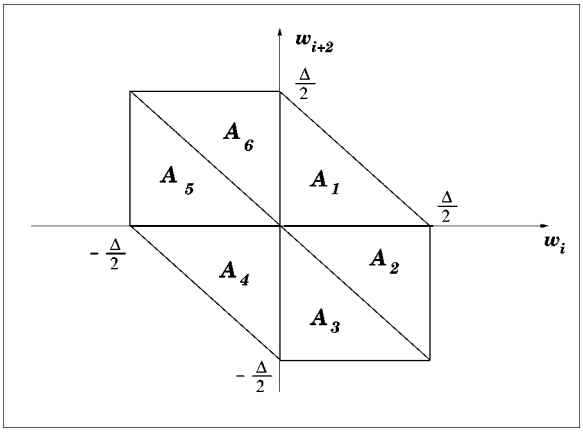

FIG. 4. Regions of the plane $(w_i = v_i - a, w_{i+2} = v_{i+2} - a)$ where the joint PDF $f_2(v_i, v_{i+2}) \neq 0$.

The most basic and important quantity characterizing correlations between the size of different Voronoi cells is given by the correlation matrix C_{ij} defined by

$$C_{ij} = \overline{(v_i - a)(v_j - a)},$$

where in this case the average is taken by using $f_2(v_i, v_j)$ [cf. Eq. (25)]. Clearly, $C_{ii} = \overline{v^2} - \overline{v}^2$. By direct calculation we have

$$C_{ij} = \begin{cases} \dfrac{\Delta^2}{24} & \text{for} \quad i = j, \\ -\dfrac{\Delta^2}{48} & \text{for} \quad i = j \pm 2, \\ 0 & \text{for} \quad i \neq j, j \pm 2. \end{cases} \tag{26}$$

We see that different Voronoi cells are either anticorrelated or uncorrelated in such a way that

$$\sum_{j=-\infty}^{+\infty} C_{ij} = 0, \tag{27}$$

i.e., positive and negative correlations must balance so that the sum of C_{ij} over j is exactly zero. Because of the strong resemblance with the basic property Eq. (9) of all the superhomogeneous point processes in arbitrary d dimensions, we expect that Eq. (27) is a general property of all superhomogeneous point processes in any dimension.

To show that this expectation is indeed true, consider a spatially stationary superhomogeneous point process in d dimensions with average density of points $n_0 > 0$. For such a point process, we know that the variance in the number of points $N(R)$ in a sphere of radius R for sufficiently large R satisfies the relation

$$\langle N^2(R) \rangle - \langle N(R) \rangle^2 \sim R^\alpha \text{ with } d - 1 \leqslant \alpha < d. \tag{28}$$

We will focus our attention on a given sufficiently large subset $\mathcal{S}$ of volume V (e.g., a sphere or an ellipsoid) and consider the number of points contained within it. The average value of this number is $\langle N(\mathcal{S}) \rangle = n_0 V$. Let us call v_i the volume of the Voronoi cell associated with point i. Since the set

of point particles is countable, we can arbitrarily label and enumerate them. By definition, $\langle v_i \rangle = 1/n_0$.

Let us now study the fluctuations of the quantity

$$U(\mathcal{S}) = \sum_{i=1}^{N(\mathcal{S})} v_i$$

under the condition of superhomogeneity. Its precise value for a single realization will fluctuate from its average value given by

$$\langle U(\mathcal{S}) \rangle = \langle N(\mathcal{S}) \rangle \langle v_i \rangle = V. \tag{29}$$

In light of Eq. (28), we can write

$$\langle |U(\mathcal{S}) - V|^2 \rangle \sim V^{\alpha/d}, \tag{30}$$

where it is to notice that $(\alpha/d) < 1$. But from Eq. (29), we can rewrite

$$\langle |U(\mathcal{S}) - V|^2 \rangle = \left\langle \sum_{i,j}^{1,N(\mathcal{S})} w_i w_j \right\rangle \sim V^{\alpha/d}, \tag{31}$$

where, as in the one-dimensional case, $w_i = v_i - 1/n_0$. This equation with $\alpha < d$ (condition of superhomogeneity), together with the fact that $N(\mathcal{S})$ grows proportionally to V and the supposed spatially stationarity of the stochastic point process, implies directly that in the limit of an infinite subset $\mathcal{S}$, we have

$$\lim_{V \to +\infty} \left\langle \sum_{j=1}^{N(\mathcal{S})} w_i w_j \right\rangle = \sum_j C_{ij} = 0, \tag{32}$$

where $C_{ij} = \langle w_i w_j \rangle$ and the last sum is extended over all of the point j of the system in the infinite volume limit. This result can be shown rigorously by various techniques, but it is sufficiently self-evident to avoid having to present the mathematical details. This result is valid for any Voronoi cell i. In fact, while the matrix $C_{ij} = \langle w_i w_j \rangle$ depends on the way we have enumerated the points, in the case of the spatially stationary point process, the sum $\Sigma_j C_{ij}$ does not depend on the enumeration. This is a quite interesting aspect of relation (32).

Therefore, in addition to Eq. (9), we have found another "sum rule" that applies to all spatially stationary superhomogeneous point processes. To check that nonsuperhomogeneous point processes do not generally satisfy Eq. (32) is a very simple task. In fact from Eq. (31) it is simple to see that if $\alpha \geqslant d$ Eq. (32) cannot hold.

VI. LARGE CELL-SIZE FLUCTUATIONS IN SUPERHOMOGENEOUS POINT PROCESSES AND VOID DISTRIBUTION

In the preceding sections, we analyzed the main properties of the one- and two-point statistics of Voronoi cells for superhomogeneous point processes. We found an important sum rule involving that the sum along any line or column of the Voronoi cells correlation vanishes for any superhomogeneous point process. In this section, we tackle two more important questions about superhomogeneous point processes: (1) Can there be infinitely large Voronoi cells, or, equivalently, infinitely large voids, for superhomogeneous point processes? (2) Is it possible to find a functional expression for void size distribution linking the probability of having a void of a certain size to the correlation properties of the superhomogeneous point process? We will see that the answers to both questions are in the affirmative.

The first question is motivated by the following facts.

(i) All the commonly known superhomogeneous point processes (lattices, quasicrystals [16], the one-component plasma [5,17], g_2-invariant processes [4], etc.) in the infinite-volume limit have only finite Voronoi cells and spherical voids.

(ii) By taking Eq. (8), for a general stochastic mass distribution (continuous or pointlike), it is possible to show [3] that, if $s(\mathbf{k}) \sim k^n$ at small k, then the wave modes which contribute essentially to create mass (i.e., number in point processes) fluctuations on large spatial scales R satisfy, $k \sim 1/R$ if $n < 1$ [and therefore $\sigma^2(R) \sim R^{d-n}$], and $k \sim k_0$ independent of R if $n \geqslant 1$ [and therefore $\sigma^2(R) \sim R^{d-1}$ for all $n \geqslant 1$]. In particular, k_0 marks the departure from the small k behavior of $s(\mathbf{k})$ to its crossover to the large k behavior; in general "shot-noise" behavior for a point process, and a rapid cutoff to zero for a continuous mass distribution. Therefore, one might surmise that, at least in the case $n > 1$, voids much larger than the inverse of this cutoff wave mode k_0 are not permitted at all. This certainly is the case for the one-dimensional model presented in the preceding section in which Voronoi cells larger than twice the original unit cell (i.e., the inverse of the average density) are not permitted.

However, more generally, we will see here that even in the case of Eq. (28) with $\alpha = d - 1$, there are superhomogeneous point processes for which we can find spherical (or spherical-like) voids (and therefore Voronoi cells) that are arbitrarily large. Moreover, and importantly, for the case of shuffled lattices, we will derive mathematical relations between the probability of applying large displacements and the probability of having a void of the same size. This will permit us to formulate an *ansatz* for the characterization of the whole class of superhomogeneous point processes in terms of the void size distribution.

With this aim, we start again from the one-dimensional regular lattice of the preceding section with lattice constant $a = 1$ and microscopic density given by Eq. (12). We then again apply to it an uncorrelated displacement field, but now we choose $p(u)$ with an unlimited tail. As already shown in Eq. (18) of the preceding section, the final point process is always superhomogeneous satisfying the condition $s(0) = 0$ for all possible $p(u)$. With the aim of simplicity but no loss of generality in the final result, we restrict the analysis to the case in which $p(-u) = p(u)$.

Let us take the segment $[0, 2R]$ (i.e., the one-dimensional sphere of radius R) with $R \gg a = 1$, and ask for the probability $W(R)$ that after the application of the displacement field no point is contained in it. Clearly, $W(R)$ can be identified also with the probability that a randomly chosen void has a radius larger than R. Therefore, $\omega(R) = -dW(R)/dR$ gives the PDF of the size (i.e., radius) of the voids.

Given a point particle initially at the lattice position m, it is simple to show that the probability $w_m(R)$, after the displacement u, outside of the segment $[0,2R]$, is

$$w_m(R) = 1 - \phi(-m) + \phi(-m+2R), \tag{33}$$

where

$$\phi(x) = \int_x^{+\infty} du\, p(u). \tag{34}$$

Note that because $p(u)$ is integrable over all the space,

$$\lim_{x\to+\infty} \phi(x) = 0 \text{ and } \lim_{x\to-\infty} \phi(x) = 1 \tag{35}$$

in any case. Since $W(R)$ is the probability that *all* of the points in the system are outside of the segment $[0,2R]$ after the displacements, we can write

$$W(R) = \prod_{m=-\infty}^{+\infty} [1 - \phi(-m) + \phi(-m+2R)]. \tag{36}$$

In this equation, we can distinguish between two multiplicative contributions by writing

$$W(R) = W_1(R) W_2(R).$$

The former contribution $W_1(R)$ is given by the points initially outside the segment $[0,2R]$, and the latter $W_2(R)$ by those initially inside it. We show that the large-R behavior of $W(R)$ is determined essentially by this second contribution.

(1) Let us consider the first contribution

$$W_1(R) = \prod_{m<0, m>2R} [1 - \phi(-m) + \phi(-m+2R)]. \tag{37}$$

Because of the discrete translational invariance of the initial configuration, $W_1(R)$ can be rewritten as

$$W_1(R) = \left[\prod_{m=1}^{+\infty} [1 - \phi(m) + \phi(m+2R)]\right]^2 = \exp\left[2\sum_{m=1}^{+\infty} \ln[1 - \phi(m) + \phi(m+2R)]\right]. \tag{38}$$

For any finite value of R, the convergence properties of the series

$$\sum_{m=1}^{+\infty} \ln[1 - \phi(m) + \phi(m+2R)] \tag{39}$$

are given by the large-m behavior of $\ln[1-\phi(m)+\phi(m+2R)]$. Because of Eq. (35) we can say that for sufficiently large m

$$\ln[1 - \phi(m) + \phi(m+2R)] \simeq -\phi(m) + \phi(m+2R).$$

At this point we must distinguish between two subcases.

(i) The PDF $p(u)$ is such that

$$\int_{-\infty}^{+\infty} du|u|p(u) < +\infty.$$

In this case,

$$\lim_{x\to+\infty} x\phi(x) = 0.$$

This implies that

$$\sum_{m=1}^{+\infty} \phi(m) < +\infty$$

and so $\Sigma_{m=1}^{+\infty}\ln[1-\phi(m)]$ will do. Therefore, to lowest order in $1/R$, we can neglect $\phi(m+2R)$ with respect to $\phi(m)$ in Eq. (38) and write

$$W_1(R) = p_0 > 0, \tag{40}$$

where $p_0 = \exp\{2\Sigma_{m=1}^{+\infty}\ln[1-\phi(m)]\}$. Corrections to Eq. (40) vanish for $R\to+\infty$.

(ii) If the PDF $p(u)$ is such that

$$\int_{-\infty}^{+\infty} du|u|p(u) = +\infty,$$

i.e., if $p(u) = Bu^{-\beta-1}$ with $0<\beta\leqslant 1$ for sufficiently large u, then

$$\lim_{x\to+\infty} x\phi(x) = +\infty.$$

This implies that

$$\sum_{m=1}^{+\infty} \phi(m) = +\infty,$$

being $\phi(m) \simeq (B/\beta)m^{-\beta}$ for large m. At any rate, the convergence of Eq. (39), for any finite R, is still ensured by the following observation. In the limit $m \gg 2R$, we can write

$$\ln[1 - \phi(m) + \phi(m+2R)] \simeq -\phi(m) + \phi(m+2R) \simeq -2B\, Rm^{-\beta-1},$$

which guarantees the convergence of Eq. (39). This implies that for large R, $W_1(R)$ will have this main behavior

$$W_1(R) \simeq \exp[-a(\beta)R^{1-\beta}], \tag{41}$$

where $a(\beta)>0$. In particular, for $\beta=1$, we expect that $W_1(R)$ goes to zero for $R\to+\infty$ as a power law.

(2) Let us now analyze the second contribution to Eq. (36),

$$W_2(R) = \prod_{0\leqslant m\leqslant 2R} [1 - \phi(-m) + \phi(-m+2R)] = \prod_{0\leqslant n\leqslant 2R} [1 - \phi(n-2R) + \phi(n)], \tag{42}$$

where in the last step we have adopted the change of variable $n=2R-m$. Note that $n-2R\leqslant 0$ and that for $R\to+\infty$ with n fixed $\phi(n-2R)\to 1$. Using the symmetry property $p(-u)=p(u)$ of the PDF of the jumps [18], we can write

$$\phi(n-2R) = 1 - \phi(2R-n).$$

Therefore, Eq. (42) can be rewritten as

$$W_2(R) = \prod_{0\leqslant n\leqslant 2R} [\phi(2R-n)+\phi(n)] = \exp\left[\sum_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)]\right]. \quad (43)$$

In order to evaluate $\Sigma_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)]$ let us approximate the sum by an integral as follows:

$$\sum_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)] \simeq \int_0^{2R} dx\, \ln[\phi(2R-x)+\phi(x)] = 2\int_0^R dx\, \ln[\phi(2R-x)+\phi(x)]. \quad (44)$$

Since $\phi(x)$ is a decreasing function of x, we can introduce a further approximation by developing the ln in the Taylor series to the first order in $\phi(2R-x)/\phi(x)$,

$$\sum_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)] \simeq 2\int_0^R dx\left[\ln\phi(x)+\frac{\phi(2R-x)}{\phi(x)}\right]. \quad (45)$$

In general, the contribution given by the term $\phi(2R-x)/\phi(x)$ can be neglected for large R with respect to the first one. We will use this approximation to study some simple but important cases: (A) a power-law tailed $p(u)$ and (B) a generalized-exponential tailed $p(u)$.

(A) Let us consider the case in which $p(u)=B\,u^{-\beta-1}$ with $\beta>0$ for sufficiently large u. In this case for sufficiently large R one obtains

$$\int_0^R dx\, \ln\phi(x) \simeq -\beta R \ln(2R).$$

This implies in the same limit of large R,

$$\sum_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)] \simeq -2\beta R \ln(2R). \quad (46)$$

Corrections to this approximation can be neglected for large R as they are of the same or lower order than R. Finally, we can write

$$W_2(R) \sim e^{-2\beta R \ln(2R)} = (2R)^{-2\beta R}.$$

We see that for any $\beta>0$ the quantity $W_2(R)$ decreases faster than an exponential $\exp(-AR)$, and therefore this is the main contribution to the behavior of $W(R)$ for large R, i.e.,

$$W(R) \sim W_2(R) \sim e^{-2\beta R \ln(2R)} = (2R)^{-2\beta R}.$$

For $\beta\to 0$, the linear corrections in R to Eq. (46) dominate giving $W(R)\sim\exp(-AR)$. This is well understood by considering that for $\beta\to 0$ the final configuration of the point distribution will no longer be superhomogeneous, but Poissonian for which it is well known that the size of voids is exponentially distributed [6].

(B) Let us consider now the case in which $p(u) \simeq c|u|^p \exp \times[-(|u|/u_0)^\alpha]$ at sufficiently large $|u|$ (in particular $|u|>u_0$) with $u_0>0$, $\alpha>0$ and any p. Again, we can use the approximation

$$\sum_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)] \simeq 2\int_0^R dx\, \ln\,\phi(x).$$

It can be shown by different techniques that for large $x \gg u_0$,

$$\phi(x) \simeq c\frac{u_0^\alpha}{\alpha} x^{p-\alpha+1} \exp\left[-\left(\frac{x}{u_0}\right)^\alpha\right][1+o(u_0/x)].$$

Therefore, for asymptotically large R (i.e., $2R\gg\max[1,u_0]$), the dominating behavior will be

$$\sum_{0\leqslant n\leqslant 2R} \ln[\phi(2R-n)+\phi(n)] \simeq -\frac{2u_0}{\alpha+1}\left(\frac{R}{u_0}\right)^{\alpha+1},$$

which implies

$$W_2(R) \sim \exp\left[-\frac{2u_0}{\alpha+1}\left(\frac{R}{u_0}\right)^{\alpha+1}\right].$$

Since in this case $W_1(R)$ approximately does not depend on R for large R, as in the previous case, $W_2(R)$ determines the behavior of $W(R)$, i.e.,

$$W(R) \sim W_2(R) \sim \exp\left[-\frac{2u_0}{\alpha+1}\left(\frac{R}{u_0}\right)^{\alpha+1}\right].$$

From the analysis of these two examples, we expect that, if $p(u)$ is a PDF with an unlimited tail such that (as power laws and generalized exponentials) at large R,

$$\ln p(\gamma R) = \gamma' \ln p(R)[1+o(1)] \quad (47)$$

with γ, γ' two positive related constants of order 1, then the following relation holds:

$$W(R) \sim \exp[AR \ln p(R)], \quad (48)$$

where A is a suitable constant depending on the average density of points n_0 and on the details of $p(u)$. In fact, this result can be generalized to any other $p(u)$ with unlimited tail [19].

The extension of this result to higher dimensions, in which again a regular lattice is perturbed by an uncorrelated displacement field characterized by a PDF with an unlimited tail, is straightforward when the PDF of the d-dimensional displacement factorizes into a product of the PDF's of the single components $p_d(\mathbf{u})=\Pi_{i=1}^d p(u_i)$. In this case, by following the same procedure for the one-dimensional case, one can find that, given a cube of large size $2R$, the probability that it becomes completely void after the application of the displacement field is

$$W(R) \sim \exp[AR^d \ln p(R)]. \quad (49)$$

We expect that the above relation, with a suitable A, is also valid if instead of taking a cube size $2R$ we take a sufficiently compact volume (e.g., a spheroid) linear size R. The mathematical treatment in the case of isotropic displacements $p_d(\mathbf{u})=p_d(u)$ is more difficult, but we expect qualitatively the

same result. We give only a rough sketch of this treatment. Let us take a sphere of very large radius R and, as above, factorize the probability $W(R)$ that after the application of the displacements it becomes empty into the product of the probability $W_1(R)$ that all particles initially out of the sphere stay out and the probability $W_2(R)$ that all the particles initially in the sphere go out of it because of the displacements. As in the previous case, we expect that $W_2(R)$ is the dominating factor for what concerns the large R decreasing behavior of $W(R)$. This can be seen through the following arguments. In order to evaluate $W_1(R)$ at sufficiently large R, we approximate the probability that a point, initially at a distance between r and $r+\Delta r$ from the center of the sphere with $r \gg R$ and $\Delta r \ll r$, will stay out of the sphere after the displacement, as

$$1 - p_d(r)\frac{\Omega_d}{d}R^d,$$

with Ω_d the complete spherical angle in d dimensions. Now the number of these particles in the initial lattice configuration is around $n_0\Omega_d r^{d-1}\Delta r$. Therefore, by taking the product over the spherical shells of thickness Δr for radii greater than R, we can write

$$W_1(R) \simeq \prod_{\Delta r}\left(1 - p_d(r)\frac{\Omega_d}{d}R^d\right)^{n_0\Omega_d r^{d-1}\Delta r}. \qquad (50)$$

In the given limits Eq. (50) can be reapproximated as

$$W_1(R) \simeq \exp\left[-n_0\frac{\Omega_d^2}{d}R^d\int_R^{+\infty} dr\; r^{d-1}p_d(r)\right]. \qquad (51)$$

In complete analogy with the one-dimensional case, it is simple now to see that, if $p_d(u)$ decays faster than u^{-2d} at large u, then $W_1(R)$ at asymptotically large L converges to a positive constant $0<p_0<1$. Instead if $p_d(u)\sim Bu^{-d-\beta}$ at large u with $0<\beta<d$ then

$$W_1(R) \simeq \exp[-aR^{d-\beta}],$$

where the constant a can be obtained approximatively by Eq. (51). For what concerns the probability $W_2(R)$ we can say that for sure it must be smaller than the probability $P(R)$ of the following event: all the particles [whose number $N(R)$ is about $(\Omega_d/d)(R/2)^d$] within a distance $R/2$ from the center of the sphere make a displacement u larger than $R/2$. This probability $P(R)$ is [in the large hypothesis Eq. (47)] roughly given by

$$P(R) = \left(\Omega_d\int_{R/2}^{+\infty} du\; u^{d-1}p_d(u)\right)^{N(R)} \simeq \exp[CR^d \ln p_d(R)]$$

with $C>0$ appropriate and depending on d, and where we have considered the fact that, by definition, $p_d(u)$ decays faster the u^{-d} at large u. On the other hand, $W(R)$ must be larger than the probability $Q(R)$ that *all* the particles in the sphere make a jump of size larger than $2R$. By similar reasoning one can find that

$$Q(R) \simeq \exp[DR^d \ln p_d(R)]$$

with D another suitable constant depending on d. Since $p_d(R)$ decrease to zero at large R, this shows at the same time that the decaying behavior of the factor $W_2(R)$ prevails on the one of $W_1(R)$, and that again $W(R)$ must have the form given by Eq. (49).

We recall now that in general in a Poisson point process in arbitrary dimension and with average density n_0, the probability $W_P(V)$ that a given volume V is found empty of points is given by

$$W_P(V) = e^{-n_0 V}. \qquad (52)$$

For point processes that are essentially Poisson with primarily positive and short-range correlations, due to the only short-range clusterization of points, we expect a similar relation for sufficiently large voids, but with n_0 replaced by an appropriate smaller constant [6,20]. On the other hand, in "critical" point processes, because of the strong clusterization of points at all scales due to large-scale positive correlations, we expect a larger probability of finding large voids than in the Poisson one. Therefore, for superhomogeneous point processes generated by displacing the points of a d-dimensional regular lattice in an uncorrelated manner, the probability that a compact volume of sufficiently large linear size R decays with R faster than in any nonsuperhomogeneous point process.

This observation suggests the following general heuristic conclusion.

(i) A point process is superhomogeneous *if and only if* its void size distribution $W(R)$ satisfies the limit condition

$$\lim_{R\to+\infty}\frac{\ln W(R)}{R^d} = 0.$$

Moreover, the above discussion about the void distribution generated in a lattice by an uncorrelated but power-law-tailed displacement PDF suggests a second general heuristic conclusion.

(ii) A point process for which

$$\lim_{R\to+\infty}\frac{\ln W(R)}{R^d \ln R} = 0$$

not only is superhomogeneous but its power spectrum satisfies $s(\mathbf{k})\sim k^n$ at sufficiently small k with $n\geqslant 2$. However, this proposition cannot be inverted. In fact, $n\geqslant 2$ is obtained also in the case of power-law-tailed $p_d(u)$, but with a finite variance.

VII. CONCLUSIONS

Superhomogeneous stochastic point processes, and more generally superhomogeneous mass stochastic density fields, are very important mathematical models of many systems not only in material science and condensed matter physics, but also in diverse fields such as cosmology. For instance, slightly perturbed crystal lattices, quasicrystals [16], one-component plasmas [5], particular glassy systems, strictly jammed stochastic hard spheres configurations [4] can all be

seen as superhomogeneous point density fields. Cosmological models predict a spectrum of the primordial mass density perturbations of the Universe typical of superhomogeneous systems [3,17], and superhomogeneous point processes (typically perturbed lattices or glasslike particle distributions) are used as initial conditions in *n*-body simulations to study the mass collapse and the structure (e.g., galaxies and clusters of galaxies) formation problems during the history of the Universe.

Usually a point process is recognized to be superhomogeneous by studying the scaling behavior of its number fluctuation $\sigma(R)$ with respect to the distance at asymptotically large spatial scales [see Eq. (10)], or by analyzing the spatial integral of the density covariance $C(\mathbf{x})$ [see Eq. (9)] or equivalently the power spectrum $s(\mathbf{k})$ at small wave numbers [see Eq. (11)].

In this paper, we have characterized superhomogeneous systems by studying the statistical properties of the Voronoi cells and of void size distribution. It is an important achievement because the knowledge of the statistical properties of Voronoi cells is an important issue in many subjects of disordered materials. This task has been accomplished mainly with the detailed study of the so-called one-dimensional "shuffled lattice," i.e., a regular chain of particles whose particles are randomly displaced from their lattice positions with no correlations between the displacements. Inspired by the achievements obtained for these systems, we have generalized the main results to the whole class of superhomogeneous point processes in arbitrary spatial dimension.

The main results that we have obtained can be summarized as follows.

(i) For a particular subclass of one-dimensional "shuffled lattices," one and two Voronoi cell statistics have been solved exactly.

(ii) The correlation matrix C_{ij} of the Voronoi cells of *any* superhomogeneous point process satisfies a sum rule $\Sigma_j C_{ij} = 0$, which is independent of the way in which the single Voronoi cells have been labeled. This is a very important relation because it is a special property of only superhomogeneous point processes. Indeed, this sum rule is the Voronoi-cell equivalent of Eq. (9), which is the definition of a superhomogeneous point process in terms of the covariance function.

(iii) In contrast to the conventional picture of superhomogeneous systems, we have shown that arbitrarily large Voronoi cells or voids are permitted in the superhomogeneous class. This is true despite the fact that superhomogeneous point processes possess the slowest number (mass) fluctuations-length scaling relation possible for any point process.

(iv) For the most general one-dimensional shuffled lattice, we have found the asymptotic form of the void size distribution and its dependence on the "shuffling" statistics.

(v) This result for void statistics has been extended to higher dimensions, and suggests the introduction of two heuristic conditions to identify and classify any superhomogeneous point process in terms of the asymptotic behavior of the void size distribution.

This last result together with the sum rule about the correlation matrix of the Voronoi cells are the two most significant achievements of this study. The present analysis and results open the possibility for new studies on even more complex morphological characterizations of superhomogeneous point processes.

ACKNOWLEDGMENTS

A.G. thanks the Physics Department of the University "La Sapienza" of Rome (Italy) for having supported this research. S. T. gratefully acknowledges the support of the Office of Basic Energy Sciences, Department of Energy, under Grant No. DE-FG02-04ER46108.

APPENDIX

In this Appendix we give a brief derivation of Eq. (13). For a more general analysis of the effect of a stochastic displacement field on the power spectrum of a given point process see Ref. [15].

Let us call $\hat{n}_I(x)=\Sigma_{i=1}^{N}\delta(x-x_i)$ the *initial* microscopic density of a given point process, defined on the line segment $[-L/2,l/2]$, where $-L/2\leqslant x_i\leqslant L/2$ is the position of the ith point particle of the system before the application of the displacement field. Ultimately, we will take the limit $L\to\infty$. Let us also suppose we know the global average density $n_0 = \lim_{L\to\infty} N/L$ and the power spectrum $s_I(k)$ of such a point process as defined, respectively, by Eqs. (2) and (5). We now apply to each point i, independently of the others, a stochastic displacement u_i extracted from the probability density function $p(u)$. The new microscopic density will be

$$\hat{n}(x) = \sum_i \delta(x - x_i - u_i).$$

By definition, the new power spectrum $s(k)$ will be given by

$$s(k) = \lim_{L\to+\infty}\frac{1}{L}\left\langle \overline{\sum_{i,j}^{1,N} e^{-ik(x_i-x_j+u_i-u_j)}}\right\rangle - 2\pi n_0^2\delta(k), \quad \text{(A1)}$$

where $\overline{(\cdots)}$ stands for the average over all the possible realizations of the displacement field for a given realization of the *initial* point process, and $\langle\cdots\rangle$ stands for the ensemble average over all the possible realizations of the initial point process. In our hypothesis the displacement field and the point process are considered statistically independent, and hence the two averages commute and they can be taken in an arbitrary order. We will first take the average over the displacements by separating the diagonal contribution from the nondiagonal one in the double sum of Eq. (A1),

$$\overline{\sum_{i,j}^{1,N} e^{-ik(x_i-x_j+u_i-u_j)}} = N + |\tilde{p}(k)|^2 {\sum_{i,j}^{1,N}}' e^{-ik(x_i-x_j)},$$

where $\tilde{p}(k)$ is defined by Eq. (14) and $\Sigma'_{i,j}$ means the sum over all $i=1,\ldots,N$ and $j=1,\ldots,N$ with $i\neq j$. Therefore, we can rewrite Eq. (A1) as

$$s(k) = \lim_{L\to+\infty}\left[\frac{N}{L}(1-|\tilde{p}(k)|^2) + \frac{|\tilde{p}(k)|^2}{L}\left\langle \sum_{i,j}^{1,N} e^{-ik(x_i-x_j)}\right\rangle\right] - 2\pi n_0^2\delta(k), \tag{A2}$$

where we have added and subtracted the term $(N/L)|\tilde{p}(k)|^2 = \Sigma_{i=1}^{N}[|\tilde{p}(k)|^2/L]$ in order to complete the double sum. Equation (13) is recovered by noticing that the following relations hold:

$$\tilde{p}(0) = 1,$$

$$\lim_{L\to+\infty}\frac{N}{L} = n_0,$$

$$\lim_{L\to+\infty}\frac{1}{L}\left\langle \sum_{i,j}^{1,N} e^{-ik(x_i-x_j)}\right\rangle - 2\pi n_0^2\delta(k) = s_I(k).$$

The extension to higher spatial dimensions is obvious.

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[18] The same arguments can be simply generalized to include asymmetric $p(u)$.

[19] It is simple to show that if we have an unlimited tail that is faster than any generalized decreasing exponential [e.g., $p(R) \sim e^{-e^R}$] , Eq. (48) must be replaced by $W(R) \sim \exp[AR \ln p(BR)]$ with $B>0$ of order 1, which is the general result for any unlimited tailed $p(u)$.

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